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Quasi-classical results on supersymmetric quantum mechanics

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1 Introduction

In 1976 Nicolai [1] introduced supersymmetric (SUSY) quantum mechanics as $(0+1)$-dimensional version of SUSY quantum field theories [2]. In particular, with Witten's model [3] SUSY quantum mechanics attracted much attention in the last two decades. Nowadays SUSY quantum mechanics is widely used as a tool in various branches of theoretical physics [4]. The aim of this contribution is to communicate some of the recent quasi-classical results on Witten's model of SUSY quantum mechanics.

In the next section we will review the concept of SUSY quantum mechanics with particular emphasis on Witten's model. Section 3 deals with recent developments made in quasi-classical approximations for this model. A new supersymmetric quasi-classical quantization condition is given and compared with the standard WKB approach. Finally, section 4 presents some numerical investigations for a half-line problem with broken SUSY.

2 Nicolai's SUSY quantum mechanics and Witten's model

According to Nicolai [1] a quantum mechanical system characterized by a self-adjoint Hamiltonian H is called *supersymmetric* if there exists a *supercharge* operator Q obeying the following anticommutation relations:

$$\{Q,Q^\dagger\} = H\,, \quad \{Q,Q\} = 0 = \{Q^\dagger,Q^\dagger\}\,. \tag{1}$$

An immediate consequence of these relations is the conservation of the supercharge $[H,Q] = 0 = [H,Q^\dagger]$ and the non-negativity of the Hamiltonian $H \geq 0$. Another consequence of SUSY is the existence of a self-adjoint grading operator

$$W := \frac{2}{H}QQ^\dagger - 1\,, \tag{2}$$

which obeys the following relations

$$W^2 = 1\,, \quad [W,H] = 0\,, \quad \{W,Q\} = 0 = \{W,Q^\dagger\}\,. \tag{3}$$

Note that W is only defined on the orthogonal complement of ker H. However, in many cases it may be extended to a well-defined operator on the full Hilbert space. The eigenspaces of W corresponding to the eigenvalues $+1$ and -1 provide a natural Z_2-grading of

the Hilbert space. With the help of $[W,H] = 0$ one can also easily show that the strictly positive eigenvalues $E > 0$ of H are pairwise degenerate,

$$H|\psi_E^\pm\rangle = E|\psi_E^\pm\rangle\,, \quad W|\psi_E^\pm\rangle = \pm|\psi_E^\pm\rangle\,, \tag{4}$$

where the two energy eigenstates are related by the SUSY transformations:

$$|\psi_E^-\rangle = \frac{1}{\sqrt{E}}Q^\dagger|\psi_E^+\rangle\,, \quad |\psi_E^+\rangle = \frac{1}{\sqrt{E}}Q|\psi_E^-\rangle\,. \tag{5}$$

In 1981 Witten [3] introduced a simple but non-trivial model of SUSY quantum mechanics which is defined on the Hilbert space $\mathcal{H} := L^2(\mathbb{R}) \otimes \mathbb{C}^2$. That is, it characterizes a spin-$\frac{1}{2}$-like point mass m moving along the real line $\mathbb{R}$. The supercharge operator is given by

$$Q := \frac{\hbar}{\sqrt{2m}}\left(\frac{\partial}{\partial x} + \Phi(x)\right) \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\,, \tag{6}$$

where the so-called *SUSY potential* $\Phi : \mathbb{R} \to \mathbb{R}$ is assumed to be continuously differentiable. Obviously the supercharge obeys the second anticommutation relation given in (1). The supersymmetric Hamiltonian is then defined via the first relation,

$$H := \{Q,Q^\dagger\} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}\,, \tag{7}$$

where we have introduced the partner Hamiltonians

$$H_\pm := \frac{\hbar^2}{2m}\left[-\frac{\partial^2}{\partial x^2} + \Phi^2(x) \pm \Phi'(x)\right] \geq 0\,, \tag{8}$$

which are standard Schrödinger operators acting on $L^2(\mathbb{R})$. As an aside let us mention that the grading operator (2) in this model is diagonal $W = 1 \otimes \mathrm{diag}(1,-1)$.

The SUSY of a quantum system is said to be a good symmetry (good SUSY) if the ground-state energy of H vanishes, inf spec $H = 0$. In the other case, inf spec $H > 0$, SUSY is said to be broken. For good SUSY the ground state of H either belongs to H_+ or H_- and is given by

$$\varphi_0^\pm(x) = \varphi_0^\pm(0)\exp\left\{\pm\int_0^x \mathrm{d}z\,\Phi(z)\right\}\,. \tag{9}$$

Obviously, the behavior of the SUSY potential at infinity provides sufficient information to decide whether SUSY is good or broken. To be more explicit, let us introduce the *Witten index*

$$\Delta := \text{dim ker } H_- - \text{dim ker } H_+ = \tfrac{1}{2}[\text{sgn }\Phi(+\infty) - \text{sgn }\Phi(-\infty)]\,, \tag{10}$$

which only depends on the sign of the SUSY potential at infinity. Hence, for good SUSY we have $\Delta = \pm 1$ with the ground state belonging to $H_\mp$. For broken SUSY we have

$\Delta = 0$. Note that the Witten index may also be given by $\Delta = -\mathrm{Tr}\,(W \exp\{-\beta H\})$ which is independent of the regularization parameter $\beta > 0$ because of the following remarks.

Let us denoted with $E_n^{\pm}$, $n = 0, 1, 2, \ldots$, the ordered set of discrete eigenvalues of $H_{\pm}$, that is, $E_n^{\pm} < E_{n+1}^{\pm}$. Then because of the SUSY transformation (5) the strictly positive eigenvalues of H_+ and H_- are identical and hence, the spectral properties of $H_{\pm}$ may be summarized as follows:

$$\begin{array}{lcl} \Delta = \pm 1 & : & E_n^{\pm} = E_{n+1}^{\mp} > 0\,, \quad E_0^{\mp} = 0\,, \\ \Delta = 0 & : & E_n^{-} = E_n^{+} > 0\,. \end{array} \tag{11}$$

3 A novel quasi-classical approximation

In 1985 Comtet, Bandrauk and Campbell [5] suggested a modified version of the WKB quantization condition which is applicable to Witten's model with good SUSY. Recently [6,7], within a modified stationary-phase approximation for the path-integral ansatz of Witten-like SUSY models, a more general so-called *quasi-classical supersymmetric* (qc-SUSY) quantization condition has been obtained, which is also applicable to cases with broken SUSY. This qc-SUSY formula reads (see [6,7,4] for details)

$$\boxed{\int_{x_L}^{x_R} \mathrm{d}x \sqrt{E/\tfrac{\hbar^2}{2m} - \Phi^2(x)} = \pi\left(n + \frac{1}{2} \pm \frac{\Delta}{2}\right)}\,, \tag{12}$$

where $x_L < x_R$ denote the left and right turning points of the so-called quasi-classical path [4], that is, $\Phi^2(x_{L/R}) = E/\frac{\hbar^2}{2m}$. In (12) the plus and minus sign has to be taken if one is interested in an approximate spectrum of H_+ and H_-, respectively.

The qc-SUSY quantization (12) condition has some remarkable properties. In the case of good SUSY, that is, $\Delta = \pm 1$, it leads to the exact ground-state energy $E_0^{\mp} = 0$. For all other cases, (12) gives in general approximate energy eigenvalues $E_n^{\pm}$, which nevertheless obey the SUSY structure (11). Furthermore and most strikingly (12) leads to the exact bound-state spectrum of all so-called shape invariant systems. Shape-invariant Hamiltonians are those which allow for a (Schrödinger-Infeld-Hull) factorization. In other words, their eigenvalue problem is exactly solvable via algebraic methods [4]. None of the above mentioned properties do apply to the usual WKB approximation

$$\int_{q_L^{\pm}}^{q_R^{\pm}} \mathrm{d}x \sqrt{E/\tfrac{\hbar^2}{2m} - \Phi^2(x) \mp \Phi'(x)} = \pi\left(n + \frac{1}{2}\right)\,, \tag{13}$$

where $q_L^{\pm} < q_R^{\pm}$ are the classical turning points, $\Phi^2(q_{L/R}^{\pm}) \pm \Phi'(q_{L/R}^{\pm}) = E/\frac{\hbar^2}{2m}$.

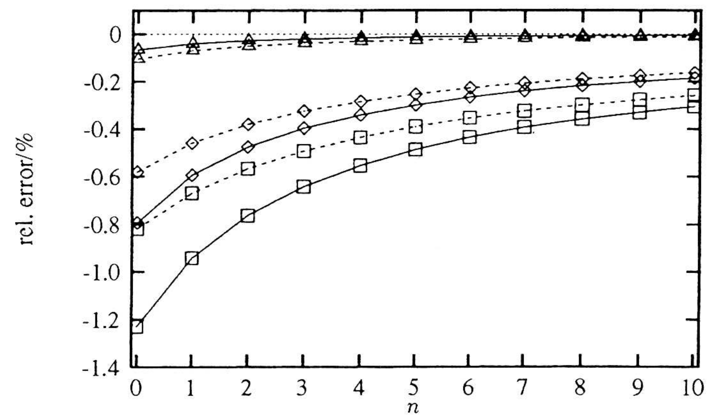

Figure 1: Relative errors (in %) of energy eigenvalues obtained via the qc-SUSY approximation (△) and the WKB approximation for V_+ (◇) and V_- (□) given in (14). For the potential parameters see the text.

4 Some numerical results

Due to the fact, that the qc-SUSY approximation (12) gives the exact spectra for the exactly solvable shape-invariant potentials one might expect that it will also be superior for not exactly solvable systems. These kind of questions have been considered by several authors. For example, in [8,7,4] the class of systems characterized by power SUSY potentials of the form $\Phi(x) = ax^d$, $a > 0$, $d = 1, 2, 3, \ldots$, is investigated with the result that the qc-SUSY approximation is at least as good as but often even better than the standard WKB approximation. In particular, for the case of power SUSY potentials with broken SUSY it has been found [7] that the qc-SUSY formula always overestimates the exact energy eigenvalue, whereas the WKB formula leads to an underestimation. This observation has also been made for non-polynomial systems on the real line [7,4]. However, for radial problems, that is, those on the half line with broken SUSY, it has recently been shown that this observed ordering of the approximate energy levels is no longer true [9].

As an example for such half-line problems ($r > 0$) with broken SUSY let us consider the SUSY potential $\Phi(r) = ar + b + c/r$, $a > 0$, $b \geq -\sqrt{ac}$, $c \geq 1$, giving rise to the partner potentials

$$V_{\pm}(r) = \frac{\hbar^2}{2m}\left[a^2r^2 + 2abr + \frac{2bc}{r} + \frac{c(c \mp 1)}{r^2} + (b^2 + 2ac \pm a)\right]\,. \tag{14}$$

Note that for $b = 0$ the partner potentials (14) reduce to those of the radial harmonic oscillator, which is a shape-invariant system.

In figure 1 we compare the approximate energy eigenvalues obtained via the qc-SUSY and WKB formula with the exact eigenvalues of $H_\pm = \frac{p^2}{2m} + V_\pm$ for two sets of parameters. Note that figure 1 shows the relative errors for the eleven lowest energy eigenvalues. The parameters are chosen to $a = 1.5$, $b = 0.5$, $c = 2$ (solid line) and $a = 1$, $b = 1$, $c = 2$ (dashed line). The numerical results clearly show that the qc-SUSY approximation is about one order of magnitude better than the WKB approximation for V_+ and V_-. For further numerical results on this system and other radial problems with broken SUSY we refer to [9].

References

[1] H. Nicolai, "Supersymmetry and spin systems", *J. Phys.* **A9**, 1497–1506 (1976).

[2] J. Wess and B. Zumino, "Supergauge transformations in four dimension", *Nucl. Phys.* **B70**, 39–50 (1974).
J. Wess and B. Zumino, "Supergauge invariant extension of quantum electrodynamics", *Nucl. Phys.* **B78**, 1–13 (1974).

[3] E. Witten, "Dynamical breaking of supersymmetry", *Nucl. Phys.* **B188**, 513–554 (1981).

[4] G. Junker, "Supersymmetric Methods in Quantum and Statistical Physics", Texts and Monographs in Physics, Springer-Verlag, Berlin, 1996.

[5] A. Comtet, A.D. Bandrauk and D.K. Campbell, "Exactness of semiclassical bound state energies for supersymmetric quantum mechanics", *Phys. Lett.* **150B**, 159–162 (1985).

[6] A. Inomata and G. Junker, "Quasi-classical approach to path integrals in supersymmetric quantum mechanics", in *Lectures on Path Integration: Trieste 1991*, (Eds. H.A. Cerdeira, S. Lundqvist, D. Mugnai, A. Ranfagni, V. Sa-yakanit and L.S. Schulman), p. 460–482, World Scientific, Singapore, 1993.

[7] A. Inomata and G. Junker, "Quasiclassical path-integral approach to supersymmetric quantum mechanics", *Phys. Rev.* **A50**, 3638–3649 (1994).

[8] A. Khare, "How good is the supersymmetry-inspired WKB quantization condition?", *Phys. Lett.* **161B**, 131–135 (1985).

[9] G. Junker, P. Roy and Y.P. Varshni, "Quasi-classical investigation of non-polynomial central potentials with broken supersymmetry", Erlangen preprint 1996.